

On isomorphism of tensor powers of ergodic flows

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Abstract

The following question due to Thouvenot is well-known in ergodic theory. Let S and T be automorphisms of a probability space and $S \otimes S$ be isomorphic to $T \otimes T$. Will S and T be isomorphic? Our note contains a simple answer to this question and a generalization of Kulaga's result on the corresponding isomorphism within a class of flows. We show that the isomorphism of weakly mixing flows $S_t \otimes S_t$ and $T_t \otimes T_t$ implies the isomorphism of the flows S_t and T_t , if one of them has an integral weak limit.

1 Introduction

The work is devoted to the problem of the isomorphism of two dynamical systems under the condition that their tensor powers are isomorphic. By isomorphisms we mind a conjugation by an invertible measure-preserving transformation (an automorphism). In [1] it was proved that for generic transformations, the isomorphism of their tensor powers implies the isomorphism of the transformations themselves. From the results of [3,4] one can deduce a similar result for typical actions of multi-parameter flows.

We generalize one of the facts from [5] on the isomorphism of flows and give a negative response to the mentioned question in the situation where systems are not required to both be ergodic. We first start with these examples.

Counterexamples. Let's consider an irrational shift of the circle $X = \mathbf{R}/\mathbf{Z}$. It preserves the Lebesgue measure μ and it is ergodic. The ergodicity means that any R -invariant measurable function on X has to be constant.

The torus $X \times X$ is stratified into $R \otimes R$ -invariant circles, where $R \otimes R(x, y) = (Rx, Ry)$. The transformation $R \otimes R$ acts on each circle as the original rotation R .

An isomorphism of the transformations $R \otimes R$ and $I \otimes R$, where I denotes the identity transformation, is given by the map

$$(x, y) \rightarrow \left(x - y, \frac{x + y}{2}\right).$$

Setting $S = I \otimes R$ and $T = R$, we get: $S \otimes S$ and $T \otimes T$ are isomorphic. Indeed, the transformation $I \otimes R \otimes I \otimes R$ is isomorphic to $I \otimes I \otimes I \otimes R$, hence, it is isomorphic to the transformation $I \otimes R$. However the non-ergodic transformation S is not isomorphic to ergodic T .

Similar counterexamples could be obtained with arbitrary ergodic transformation R with purely discrete spectrum. In the case of flows, let us consider R_t , an ergodic torus winding, and put $T_t = R_t$ and $S_t = I \otimes R_t$. All tensor powers of the flows T_t, S_t are isomorphic between themselves, but the flows T_t, S_t are not isomorphic, since the first flow is ergodic, and the second is not.

Thouvenot's question remains open in the class of weakly mixing dynamical systems, in particular, for the flows with continuous spectrum. Counterexamples within this class, if they exist, must have, in our opinion, unusual properties. The following question is also of interest: *for which groups G there are no such counterexamples among the measure-preserving G -actions?*

Main result. In [5], in particular, it is proved that the isomorphism of flows $S_t \otimes S_t \sim T_t \otimes T_t$ implies the isomorphism of the flows S_t and T_t , if the flow T_t has a weak limit in the form $\int_{\mathbf{R}} T_a d\nu(a)$, where ν is a continuous measure on \mathbf{R} with analytical Fourier transform. We shall show that this last condition is not necessary.

Theorem. *The isomorphism of flows $S_t \otimes S_t$ and $T_t \otimes T_t$ implies the isomorphism of the flows S_t and T_t , if the flow T_t has a weak limit in the form*

$$\int_{\mathbf{R}} T_a d\nu(a),$$

where ν is a continuous measure on \mathbf{R} .

Remark. The theorem is true for any non-Dirac measure ν . The presence of pointwise components of the measure simplifies the proof (see [2]). We confine ourselves to the most interesting case.

2 Proof of Theorem

Let Φ denote the isomorphism of the measure-preserving flows $S_t \otimes S_t$ and $T_t \otimes T_t$. Subsequently, the transformations and the operators on L_2 corresponding to them are denoted identically. Instead of equations of the form

$$S_t \otimes S_t = \Phi(T_t \otimes T_t)\Phi^{-1}$$

below we will write

$$S_t \otimes S_t =_{\Phi} T_t \otimes T_t.$$

For some sequence t_i , we have

$$T_{t_i} \rightarrow \int_{\mathbf{R}} T_a d\nu(a)$$

(here and below we consider the weak operator convergence). For some Markov operator Q we have

$$S_{t_i} \rightarrow Q$$

and the equality

$$Q \otimes Q =_{\Phi} \int \int T_a \otimes T_b d\nu(a) d\nu(b).$$

Markov operator Q commutes with the flow S_t . Recall that Markov operators preserve the non-negativity of functions and send the constants to themselves. Let us consider ρ , the measure on $X \times X$, defined by the relation

$$\rho(A \times B) = (Q\chi_A, \chi_B)$$

for all measurable sets A, B . It is invariant with respect to the flow $S_t \otimes S_t$ and has the standard marginals: its projections onto the factors in $X \times X$ are μ . In ergodic theory such measures are called self-joinings.

The measure ρ decomposes into ergodic with respect to $S_t \otimes S_t$ components $\rho_c, c \in C$,

$$\rho = \int_C \rho_c \sigma(c).$$

Let us show the ergodicity (almost surely) of the measures $\rho_c \times \rho_{c'}$ with respect to the transformation $S_t \otimes S_t \otimes S_t \otimes S_t$. Recall the well-known fact of the spectral theory of ergodic transformations and flows: the tensor product of two systems is not ergodic iff these systems have the same eigenvalue different from 1. This eigenvalue is inherited by ergodic components of this tensor product.

Let us consider the self-joining of η , corresponding to the operator

$$\int_R \int_R T_a \otimes T_b d\nu(a) d\nu(b).$$

The ergodic components of η are sitting on the graphs of the transformations $T_a \otimes T_b$. The flow $(T_t \otimes T_t \otimes T_t \otimes T_t, \eta)$ is isomorphic to the flow $(S_t \otimes S_t \otimes S_t \otimes S_t, \rho \times \rho)$. All ergodic components of the flow $(T_t \otimes T_t \otimes T_t \otimes T_t, \eta)$ are isomorphic to the weakly mixing flow $(T_t \otimes T_t, \mu \times \mu)$. Consequently, for almost all c, c' with respect to $\nu \times \nu$ the dynamical system $(S_t \otimes S_t \otimes S_t \otimes S_t, \rho_c \times \rho_{c'})$ is isomorphic via Φ to the flow $(T_t \otimes T_t, \mu \times \mu)$. Indeed, if the measures $\rho_c \times \rho_{c'}$ is non-ergodic, then the flow $(T_t \otimes T_t, \mu \times \mu)$ possesses a non-constant proper function, but it is not true.

Let Q_c denote the Markov operators, corresponding to the self-joining ρ_c . We have

$$Q = \int_C Q_c d\sigma(c),$$

$$\int_C \int_C Q_c \otimes Q_{c'} d\sigma(c) d\sigma(c') =_{\Phi} \int_R \int_R T_a \otimes T_b d\nu(a) d\nu(b).$$

The isomorphism Φ maps extreme points to extreme points, so, it maps the Markov operators $Q_c \otimes Q_{c'}$ to the operators $T_a \otimes T_b$. The isometry maps automorphisms to automorphisms, so Q_c is an automorphism commuting with the flow S_t . For some functions $c(a, b)$ and $c'(a, b)$ we obtain the equality

$$Q_{c(a,b)} \otimes Q_{c'(a,b)} =_{\Phi} T_a \otimes T_b$$

for almost all a, b with respect to the measure ν . From $S_t \otimes S_t =_{\Phi} T_t \otimes T_t$, we get now

$$S_{-b} Q_{c(a,b)} \otimes S_{-b} Q_{c'(a,b)} =_{\Phi} T_{a-b} \otimes I. \quad (*)$$

This equality holds for some different numbers a, b . We pick them.

Case 1. If one of the factors in the product $S_{-b} Q_c \otimes S_{-b} Q_{c'}$ is the identity operator, and the other is ergodic, then Theorem is proved. Indeed, in this case the isomorphism Φ maps the coordinate algebra, corresponding to the identity operator from $T_{a-b} \otimes I$, to one of the coordinate algebras. It remains to show that other cases cannot be realized.

Case 2. Let $S_{-b} Q_c$ and $S_{-b} Q_{c'}$ both be ergodic. They will have a continuous spectrum. Indeed, the automorphism T_{a-b} has continuous spectrum, therefore,

the operator $T_{a-b} \otimes I$ has no eigenvalues, except 1. So the product $S_{-b}Q_c \otimes S_{-b}Q_{c'}$ has continuous spectrum, it cannot be isomorphic to $T_{a-b} \otimes I$.

Case 3. Let the automorphisms of $S_{-b}Q_c$ and $S_{-b}Q_{c'}$ both be not ergodic and each of them is not the identity automorphism.

Following Rokhlin, we represent them in the form of skew products over identity transformations.

From (*) we see that the automorphism T_{a-b} can be represented as $T' \otimes T''$, where T' and T'' are isomorphic to the ergodic components of the automorphisms $S_{-b}Q_c$ and $S_{-b}Q_{c'}$, respectively. We will show that this case is impossible.

Put $d = b - a \neq 0$. We have

$$T_{t_i} \rightarrow \int_{\mathbf{R}} T_a d\nu(a).$$

We find some integer sequence k_i such that $t_i = k_i d + r_i$, $r_i \rightarrow r \in [0, d)$, and

$$T_{k_i d} \rightarrow J = \int_{\mathbf{R}} T_u d\nu'(u),$$

where ν' is r -shift of the measure ν . The limit operator J (as well as all operators $T_{k_i d}$) commutes with the orthogonal projection P onto the space of T' -factor. Let us rewrite the equality

$$PJ = JP$$

in the form

$$\int_{\mathbf{R}} PT_u d\nu'(u) = \int_{\mathbf{R}} T_v P d\nu'(v).$$

The operators $PT_u, T_v P$ are extreme points in the convex compact set of all Markov operators, commuting with the automorphism T_d . This is equivalent to the fact that they correspond to ergodic self-joinings. The dynamical system associated with these self-joinings is isomorphic to the ergodic transformation $T' \otimes T'' \otimes T''$. The ergodicity follows from the property of the weak mixing of the factors.

Thus, for almost all u with respect to the measure ν' there is a number $v = v(u)$ such that

$$T_u P = T_v P.$$

Suppose that there exists u such that $u \neq v(u)$. Then we have

$$T_{u-v}P = T_uPT_{-u},$$

$$T_{(u-v)n}P = T_{un}PT_{-un}. \quad (**)$$

Let us represent

$$un = dm_n + s_n, \quad 0 \leq s_n \leq d,$$

and let

$$s_{n_k} \rightarrow s$$

and

$$T_{(u-v)n_k} \rightarrow \Theta,$$

where Θ is the orthoprojection onto the constants in $L_2(X, \mu)$. For the weakly mixing automorphism $T_{(u-v)}$ we find easily such a sequence n_k . Using

$$P = T_{dm}PT_{-dm}$$

from $(**)$ we get

$$\Theta = \Theta P = T_sPT_{-s},$$

$$\Theta = P,$$

but the latter contradicts the definition of the operator P .

Thus, the assumption that $u \neq v(u)$ leads to a contradiction. So $u = v(u)$, and, hence, the equality

$$PT_u = T_uP$$

holds for almost all u with respect to the measure ν' . From the continuity of the flow and the the continuity of the measure ν' it follows that the operator P commutes with all elements of the flow. Therefore, the corresponding factor-algebra of the automorphism T_d is a factor of the flow T_t as well.

The weak limits are inherited by the factors of flow, thus, from

$$T_{t_i} \rightarrow \int_{\mathbf{R}} T_u d\nu(u)$$

we get

$$\int_{\mathbf{R}} (T'_c \otimes T''_c) d\nu(c) = \int_{\mathbf{R}} T'_a d\nu(a) \otimes \int_{\mathbf{R}} T''_b d\nu(b).$$

But this equality is possible only in the case when the measure ν is concentrated at one point.

Otherwise we would have for some different a, b the equality

$$T'_c \otimes T''_c = T'_a \otimes T''_b,$$

which is impossible. Thus, the case 3 does not occur.

Case 3'. Recall that in the case 3 the automorphisms $S_{-b}Q_c$ and $S_{-b}Q_{c'}$ are considered both non-ergodic and not the identity automorphism.

Let now $S_{-b}Q_{c'}$ be ergodic, then it can play the role of T'' , and this situation is actually not different from the case 3. So, our present case is impossible due to the same reasons.

Case 4. Let one of the automorphisms $S_{-b}Q_c$ and $S_{-b}Q_{c'}$ be non-ergodic and the other be the identity operator. Applying

$$S_{b-a} \otimes S_{b-a} =_{\Phi} T_{b-a} \otimes T_{b-a},$$

from (*) we get

$$S_{-a}Q_c \otimes S_{-a}Q_{c'} =_{\Phi} I \otimes T_{b-a}.$$

One of the automorphisms $S_{-a}Q_c$, $S_{-a}Q_{c'}$ is ergodic, and the other is not. We are back to the case 1, since 3' is not possible. The theorem is proved.

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